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A Comparative Study of Stein–Minimax and Weighted Average Estimators with the Minimax Estimator

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Abstract

Parameter estimation is one of the most fundamental problems in statistical inference. The efficiency of an estimator is generally evaluated based on properties such as unbiasedness, consistency, efficiency, admissibility, and minimaxity. Classical estimators, although unbiased, may not always achieve the minimum possible risk. Stein's pioneering work demonstrated that for estimating multivariate normal means (dimension three or higher), the usual estimator can be improved through shrinkage. Since then, Stein-type minimax estimators and weighted average estimators have attracted considerable attention because of their ability to reduce mean squared error (MSE). This paper presents a comparative study of the classical minimax estimator, the Stein–Minimax estimator, and the weighted average estimator. The comparison is based on theoretical properties, quadratic risk, bias, variance, and mean squared error. The study concludes that Stein–Minimax estimators generally outperform the traditional minimax estimator under quadratic loss by shrinking estimates toward a target value, while weighted average estimators offer flexibility when prior or auxiliary information is available.

Keywords: Minimax estimator, Stein estimator, James–Stein estimator, Weighted average estimator, Mean squared error, Quadratic loss, Shrinkage estimation.

1. Introduction:

For analysing the efficiency property of any estimation procedure for the coefficients in a linear regression model, the performance criteria is either goodness of fitted model or precision of the estimates. The minimax estimation in linear models has received considerable attention. If one has prior information of unknowns parameters vector β, the resulting minimax linear estimator has a smaller quadratic risk than GLSE, a fairly extensive discussion of the problem of minimax-estimation can be found in Rao (1976), Bunke (1975), Bibby and Toutenburg (1977), Toutenburg (1975,1976).

In 1956, Charles Stein showed that when estimating the mean vector of a multivariate normal distribution with three or more dimensions under squared-error loss, the usual estimator is inadmissible. The James–Stein estimator dominates the classical estimator by shrinking observations toward a common target, thereby reducing overall risk. This discovery fundamentally changed statistical estimation theory.

Weighted average estimators represent another important class of estimators. They combine multiple estimators using predetermined or data-driven weights, balancing bias and variance to improve estimation accuracy.

Section 2 presents the model specifications and estimators. In Section 3 the properties of the estimators have been studied whereas Section 4 deals with a comparative study of these estimators. In Section 5 derivation of the theorem has been carried out.

2 Model Specification and Estimators

Let us consider the following linear regression model,

$$Y = X\beta + \sigma u\,, \tag{2.1}$$

where Y is a (n ×1) vector of n observations on the variables to be explained, X is a (n×p) matrix of n observations on p explanatory variables, β is a $(p\times 1)$ vector of regression coefficients, σ being unknown scalar and u is a $(n\times 1)$ vector of disturbances with mean zero, variance unity and measures of skewness and kurtosis are γ_1 and (γ_2 +3) respectively.

Application of least squares to (2.1) yields the ordinary least squares estimator as

$$b_o = (X'X)^{-1} X'Y\,. \tag{2.2}$$

which is well known to be the best linear unbiased estimator of β, having variance- covariance matrix,

$$V(b_o) = \sigma^2 (X'X)^{-1}\,. \tag{2.3}$$

The Stein- rule estimator of the regression coefficients from (2.1) is given by

$$b_s = \left[1 - k\,\frac{(Y - Xb_o)'(Y - Xb_o)}{b_o X'Xb_o}\right] b_o\,, \tag{2.4}$$

where k is any positive non-stochastic scalar characterizing the estimator.

The Minimax linear estimator (MILE) proposed by Kuks and Olman (1971, 1972) and Kuks (1972) is

$$b^* = \quad D^{-1}X'W^{-1}Y\,, \tag{2.5}$$

where D = $(\alpha^{-1}\sigma^2 T + C)$ and for $\alpha \to 0$, $b^* \to b_g$.

Using above defined quantities, we obtain

$$b^* = \beta + \sigma(X'W^{-1}X)^{-1}X'W^{-1}u - \sigma^2\frac{T}{\alpha}(X'W^{-1}X)^{-1}\beta$$

$$-\sigma^3\frac{T}{\alpha}X'W^{-1}u(X'W^{-1}X)^{-2} \quad . \tag{2.6}$$

The Stein-minimax estimator of the regression coefficient β is obtained by replacing b_o by b^* in (2.4) and is given by

$$b_s^* = \left[1 - k\frac{(Y - Xb^*)'(Y - Xb^*)}{b^* X'Xb^*}\right]b^*. \tag{2.7}$$

The balanced loss function proposed by Zellner (1994) is given as

$$L(\tilde{\beta},\beta) = \lambda_1(Y - X\tilde{\beta})'(Y - X\beta) + (1-\lambda_2)(\tilde{\beta} - \beta)X'X(\tilde{\beta} - \beta),$$

$$= \tilde{\beta} - \beta)'X'X(\tilde{\beta} - \beta) - 2\sigma\lambda_1 u'X(\tilde{\beta} - \beta) + \sigma^2\lambda_1 u'u\,. \tag{2.8}$$

where $0 \le \lambda_1 \le 1$.

For λ_1 =1 provides goodness of fit of the estimator $\tilde{\beta}$ whence λ_1 =0 focuses on the precision of the estimator. Any other value of λ_1 between 0 and 1 gives different weight to the goodness of fit and precision of the estimator. When balanced loss function is used, another estimator which is weighted average estimator of the minimax estimator and the Stein-minimax estimator is given by

$$\tilde{\beta} \;=\; \lambda_2\, b^* \;+\; (1-\lambda_2)\, b_s^* \;. \tag{2.9}$$

where $\lambda_2 \in (0,1)$. Clearly λ_2 = 1 yields minimax estimator and λ_2 = 0 gives the Stein- minimax estimator.

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3 Properties of the Estimators

In order to derive the above estimators, we shall use the small asymptotic theory introduced by the Kadane (1971) and define the following notations

$$P_x = X(X'W^{-1}X)^{-1}X'W^{-1}\ , \tag{3.1}$$

$$M = I - P_x . \tag{3.2}$$

Theorem 3.1: When the disturbances are small and not necessarily normal, the risk of weighted average estimator ($\tilde{\beta}$) up to order of approximation $o(\sigma^4)$ is given as

$$\begin{aligned} R(\tilde{\beta}) = & \sigma^2\left[\lambda_1 + (1-2\lambda_1)tr\,P_x\right] \\ & -\sigma^3\left[2k\frac{\gamma_1(1-\lambda_1)(1-\lambda_2)}{\beta'X'X\beta}\beta(I_n * B)e\right] - \sigma^4\left[\{(X'W^{-1}X)^{-1}\right. \\ & -\frac{T}{\alpha}(X'W^{-1}X)^{-2}\ \beta'X'X\beta - (1-2\lambda_1)\frac{T}{\alpha}(X'W^{-1}X)^{-1} - 2k(1-\lambda_2). \\ & \frac{T}{\alpha}(X'W^{-1}X)^{-1}trB\Big\} - \frac{k^2(1-\lambda_2)^2}{\beta'X'X\beta}\{trB(B). + 2trB\} \\ & -6k(1-\lambda_1)(1-\lambda_2)\{trB + 2B\}\ . \\ & \left.-\gamma_2\left\{\frac{k^2(1-\lambda_2)^2}{\beta'X'X\beta}trB(I_n * B) + 6k(1-\lambda_1)(1-\lambda_2)(I_n * B)\right\}\right]. \end{aligned} \tag{3.3}$$

where,

$$\text{B} = \left[1 - 2(1-M)W^{-1} + W^{-1}(1-M)W^{-1}\right]$$

" *' denotes Hadmard product operator and 'e' is a (n×1) vector with all elements as unity. When $\lambda_2 = 1$, we get the expression of the risk of minimax estimator under balanced loss function while $\lambda_2 = 0$, gives the risk of Stein-minimax estimator under balanced loss. These risks are given respectively as

$$R(b_s^*) = \sigma^2\left[\lambda_1 + (1-2\lambda_1)trP_x\right] - \sigma^3\left[2k\gamma_1 \frac{(1-\lambda_1)}{\beta' X' X\beta}\beta(In*B)e\right]$$

$$-\sigma^4\left[(X'W^{-1}X)^{-1} - \frac{T}{\alpha}(X'W^{-1}X)^{-2}\beta' X' X\beta\right.$$

$$-(1-2\lambda_1)\frac{T}{\alpha}(X'W^{-1}X)^{-1} - 2k\frac{T}{\alpha}(X'W^{-1}X)^{-1}trB$$

$$\left. -\frac{k^2}{\beta' X' X\beta}\{trB(B)+2trB\} - 6k(1-\lambda_1)\{trB+2B\}\right]$$

$$-\gamma_2\left\{\frac{k^2}{\beta' X' X\beta}trB(In*B) + 6k(1-\lambda_1)(In*B)\right\}. \qquad (3.4)$$

$$R(b^*) = \sigma^2\left[\lambda_1 + (1-2\lambda_1)trP_x\right] - \sigma^4\left[(X'W^{-1}X)^{-1}\right.$$
$$\left. -\frac{T}{\alpha}(X'W^{-1}X)^{-2}\beta' X' X\beta - (1-2\lambda_1)\frac{T}{\alpha}(X'W^{-1}X)^{-1}\right]. \qquad (3.5)$$

4 A Comparison.

(i) <u>Comparison between minimax estimator and weighted average estimator.</u>

Here, we compare the minimax estimator and weighted average estimator $\tilde{\beta}$ following the risk criterion, as

$$R(b^*) - R(\tilde{\beta}) = 2\sigma^3\gamma_1 k\frac{(1-\lambda_1)(1-\lambda_2)}{\beta' X' X\beta}\beta(In^* B)e$$

$$+\sigma^4\left[2k(1-\lambda_1)\frac{T}{\alpha}(X'W^{-1}X)^{-1}trB\right.$$

$$+\frac{k^2(1-\lambda_2)^2}{\beta' X' X\beta}\{trB.(B)+2trB\} + 6k(1-\lambda_1)(1-\lambda_2)\{trB+2B\}$$

$$\left. +\gamma_2\left\{\frac{k^2(1-\lambda_2)^2}{\beta' X' X\beta}trB(In^* B) + 6k(1-\lambda_1)(1-\lambda_2)(In*B)\right\}\right]. \qquad (4.1)$$

We find that up to $0(\sigma^3)$ of approximations, weighted average estimator $\tilde{\beta}$ have smaller risk than minimax estimator b^* so long as γ_1 and β have same sign, otherwise reverse holds true. If either $\gamma_1=0$ or $\lambda_1=1$ or $\lambda_2=1$ then both the estimator have same risk. We observe that weighted average estimator $\tilde{\beta}$ dominates minimax estimator b^* with respect to risk criterion if k satisfies

$$0<k<\frac{1}{\frac{\sigma^4(1-\lambda_2)^2}{\beta' X' X\beta}\{\{trB(B)+2trB\}+\gamma_2 trB(In*B)\}}\left[2\sigma^3\gamma_1\frac{(1-\lambda_1)(1-\lambda_2)}{\beta' X' X\beta}\right.$$

$$\beta(In*B)+2\sigma^4(1-\lambda_1)\frac{T}{\alpha}(X'W^{-1}X)^{-1}trB$$

$$\left.+6\sigma^4(1-\lambda_1)(1-\lambda_2)\ \{trB+2B+\gamma_2(In*B)\}\right]$$

If W=1, then it reduces to

$$0<k<\frac{1}{\frac{\sigma^4(1-\lambda_2)^2}{\beta' X' X\beta}\{\{(n-p)(n-p+2)\}+\gamma_2 trB(In*B)\}}.$$

$$.\left[2\sigma^3\gamma_1\frac{(1-\lambda_1)(1-\lambda_2)}{\beta' X' X\beta}\beta(In*B)+2\sigma^4(1-\lambda_1)\frac{T}{\alpha}(X'X)^{-1}(n-p)\right.$$

$$\left.+6\sigma^4(1-\lambda_1)(1-\lambda_2)\{(n-p)(n-p+2)+\gamma_2(In*B)\}\right].$$

and for normal distributed disturbances, then it reduces to

$$0<k<\left[2(1-\lambda_1)\frac{T}{\alpha}(X'X)^{-1}(n-p)+6(1-\lambda_1).\right.$$

$$\left.(1-\lambda_2)\{(n-p)(n-p+2)+\gamma_2(In*B)\}\right].$$

$$\left[\frac{(1-\lambda_2)^2}{\beta' X' X\beta}\{\{(n-p)(n-p+2)\}+\gamma_2 trB(In*B)\}\right]^{-1}$$

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(ii). Comparison between Stein-minimax and weighted average estimator

We compare the Stein-minimax estimator and weighted average estimator $\tilde{\beta}$ following the risk criterion, we have

$$R(b_s^*) - R(\tilde{\beta}) = 2\sigma^3\gamma_1 k\frac{(1-\lambda_1)\lambda_2}{\beta' X' X\beta}\beta(In*B)e$$

$$+\sigma^4\Bigg[2k\lambda_2\frac{T}{\alpha}(X'W^{-1}X)^{-1}trB$$

$$+\frac{k^2\lambda_2^{\,2}}{\beta' X' X\beta}\{trB.(B)+2trB\}+6k(1-\lambda_1)\lambda_2\{trB+2B\}$$

$$+\gamma_2\left\{\frac{k^2\lambda_2^{\,2}}{\beta' X' X\beta}trB(In*B)+6k(1-\lambda_1)\lambda_2(In*B)\right\}\Bigg]. \qquad (4.2)$$

We find that up to $0(\sigma^3)$ of approximations, weighted average estimator $\tilde{\beta}$ have smaller risk than the Stein-minimax estimator b_s^* so long as γ_1 and β have same sign, otherwise reverse holds true. If either γ_1 =0 or λ_1 =0 or λ_2 =1 then both the estimator have same risk. We find that weighted average estimator $\tilde{\beta}$ dominates Stein-minimax estimator if k satisfies

$$0 < k < \left(\frac{\sigma^4\lambda_2^{\,2}}{\beta' X' X\beta}\{\{trB(B)+2trB\}+\gamma_2 trB(In*B)\}\right)^{-1}$$

$$\Bigg[2\sigma^3\gamma_1\frac{(1-\lambda_1)\lambda_2}{\beta' X' X\beta}\beta(In*B)e+2\sigma^4\lambda_2\frac{T}{\alpha}.$$

$$(X'W^{-1}X)^{-1}trB+6\sigma^4(1-\lambda_1)\lambda_2\{trB+2B+\gamma_2(In*B)\}\Bigg].$$

If $W = I$, then it reduces to

$$0 < k < \left(\frac{\sigma^4\lambda_2^{\,2}}{\beta' X' X\beta}\{(n-p)(n-p+2)+\gamma_2 trB(In*B)\}\right)^{-1}$$

$$\Bigg[2\sigma^3\gamma_1\frac{(1-\lambda_1)\lambda_2}{\beta' X' X\beta}\beta(In*B)e+2\sigma^4\lambda_2\frac{T}{\alpha}(X' X)^{-1}(n-p)$$

$$+6\sigma^4(1-\lambda_1)\lambda_2\{(n-p)(n-p+2)+\gamma_2(In*B)\}]$$

and for normal distributed disturbances, then it reduces to

$$0 < \mathrm{k} < \frac{2\lambda_2\frac{T}{\alpha}(X'X)^{-1}(n-p)+6(1-\lambda_1)\lambda_2\{(n-p)(n-p+2)\}}{\frac{\lambda_2{}^2}{\beta'X'X\beta}\{(n-p)(n-p+2)\}}.$$

5. Proof of the Theorem

Theorem 3.1

Using (2.1), (2.2) and (2.5) in (2.6), we obtain

$$b^* = \beta+\sigma(X'W^{-1}X)^{-1}X'W^{-1}u+\sigma^2\frac{T}{\alpha}(X'W^{-1}X)^{-1}\beta$$

$$-\sigma^3\frac{T}{\alpha}X'W^{-1}U(X'W^{-1}X)^{-2} \quad . \tag{5.1}$$

Using (5.1) up to order o (σ^2) we have

$$b^{*'}X'Xb^* = \beta'X'X\beta+2\sigma\beta'X'u-2\sigma^2\frac{T}{\alpha}(X'W^{-1}X)\beta'X'X\beta$$

$$+\sigma^2u(X'W^{-1}X)W^{-1}X'Xu.$$

$$b^{*'}X'Xb^* = \beta'X'X\beta[I-\frac{2\sigma\beta'X'u}{\beta'X'X\beta}+\sigma^2\{\frac{X'W^{-1}u'Xu}{\beta'X'X\beta}-\frac{2T}{\alpha}\}(X'W^{-1}X)]. \tag{5.2}$$

$$\left[b^{*'}X'Xb^*\right]^{-1} = \frac{1}{\beta'X'X\beta}\left[1-\frac{2\sigma\beta'X'u}{\beta'X'X\beta}+\quad\sigma^2\left(\frac{X'W^{-1}u'Xu}{\beta'X'X\beta}-\frac{2T}{\alpha}\right)(X'W^{-1}X)^{-1}\right].$$

Again using (2.1) and (5.1), we obtain

$$(Y - Xb^*) = \sigma Au - \sigma^2 \frac{T}{\alpha} X(X'W^{-1}X)^{-1}\beta, \tag{5.3}$$

Up to order $o(\sigma^2)$, where A is not idempotent and is given by $A = 1 - W^{-1} + MW^{-1}$ and W is symmetric and positive definite matrix.

Using (5.3), we obtain

$$(Y - Xb^*)'(Y - Xb^*) = \sigma^2 u'A'Au - \sigma^3 \frac{T'}{\alpha} \beta'(X'W^{-1}X)X'$$

$$+ \sigma^4 \frac{T'T}{\alpha^2} \beta'(X'W^{-1}X)X'X(X'W^{-1}X)\beta. \tag{5.4}$$

Suppose T = $X'X$, then

$$(Y - Xb^*)'(Y - Xb^*) = \sigma^2 u'Bu + \sigma^4 \frac{1}{\alpha^2} \beta'(X'W^{-1}X)^{-2}(X'X)^2 \beta. \tag{5.5}$$

Using (5.1), (5.2) and (5.3) in (2.7) ,we obtain

$$b_S^* = \beta + \sigma(X'W^{-1}X)^{-1} X'W^{-1}u - \sigma^2 \left[\frac{T}{\alpha}(X'W^{-1}X)^{-1}\beta + k \frac{u'Bu}{\beta'X'X\beta}\beta \right]$$

$$- \sigma^3 \left[\frac{T}{\alpha} X'W^{-1}u(X'W^{-1}X)^{-2} + k \frac{u'Bu}{\beta'X'X\beta}(X'W^{-1}X)^{-1}X'W^{-1}u \right.$$

$$+ \frac{2k}{\alpha}\beta'\left((X'X)^{-1} - (X'W^{-1}X)^{-2}\right)\frac{TX'u\beta}{\beta'X'X\beta}$$

$$\left. - 2k \frac{u'Bu}{\beta'X'X\beta}\beta'X'u.\beta \right] - \frac{\sigma^4}{\beta'X'X\beta}\left[k \frac{u'Bu}{\beta'X'X\beta}\beta\left(X'W^{-1}X.X'u - \frac{2T}{\alpha}\beta'X'X\beta \right) \right.$$

$$+ \frac{2k}{\beta'X'X\beta}\left\{ \beta'X'u\left(u'Bu(X'W^{-1}X)^{-1}X'W^{-1}u\right) \right.$$

$$\left. + \frac{2}{\alpha}\beta'\left((X'X)^{-1} - (X'W^{-1}X)^{-2}\right)\right\} TX'u\beta$$

$$+\frac{k}{\alpha}(X'W^{-1}X)^{-1}\left\{\beta\left(Tu'Bu-\frac{\beta}{\alpha}\left((X'W^{-1}X)^{-1}(X'X)^{2}\right)\beta\right)\right\}$$

$$+2k\beta'\left\{(X'X)^{-1}-(X'W^{-1}X)^{-1}\right\}TX'u'X'W^{-1}u\}\Big]. \qquad (5.6)$$

By substituting (5.1) and (5.6) in (2.9), we get

$$\tilde{\beta} = \beta+\sigma(X'W^{-1}X)^{-1}X'W^{-1}u - \sigma^{2}\left[\frac{T}{\alpha}(X'W^{-1}X)^{-1}\beta\right]$$

$$-\sigma^{3}\frac{T}{\alpha}(X'W^{-1}X)^{-2}X'u-k(1-\lambda_{2})\left[\frac{\sigma^{2}u'Bu}{\beta'X'X\beta}+\frac{\sigma^{3}}{\beta'X'X\beta}.\ u'Bu(XW^{-1}X)^{-1}X'W^{-1}u\right.$$

$$\left.+\frac{2}{\alpha}\beta'\left((X'X)^{-1}-(XW^{-1}X)^{-2}\right)TX'u\beta\ -\frac{2u'Bu}{\beta'X'X\beta}\beta'X'u.\beta\Big)\right]. \qquad (5.7)$$

Using (5.7) in (2.8) we get

$$L(\tilde{\beta},\beta)=\sigma^{2}\phi_{1}+\sigma^{3}\phi_{2}+\sigma^{4}\phi_{3}. \qquad (5.8)$$

where,

$$\phi_{1} = \lambda_{1}u'u+u'X(X'W^{-1}X)^{-1}X'W^{-1}u\left(X'W^{-1}(X'W^{-1}X)^{-1}-2\lambda_{1}\right). \qquad (5.9)$$

$$\phi_{2} = -\left[2X(X'W^{-1}X)^{-1}X'W^{-1}u\left\{\frac{T'}{\alpha}(X'W^{-1}X)^{-1}X'+k(1-\lambda_{1})\right.\right.$$

$$\left.\left.\frac{u'Bu}{\beta'X'X\beta}\beta'X'\ \right\}-2\lambda_{1}\left\{\frac{T}{\alpha}X(X'W^{-1}X)^{-1}\beta u'+k(1-\lambda_{2})\frac{u'Bu}{\beta'X'X\beta}u'X\beta\right\}\right]. \qquad (5.10)$$

$$\phi_3 = \left[\left\{\frac{T'T}{\alpha^2}\beta' X' X\beta - \frac{2T}{\alpha} X'W^{-1}uu'W^{-1}X(X'W^{-1}X)^{-1}X'X\right\}(X'W^{-1}X)^{-2}\right.$$

$$-2\{u'W^{-1}X(X'W^{-1}X)^{-1}\}\left\{u'\beta u'(X'W^{-1}X)^{-1}XX'W^{-1}u\frac{2\beta' X'u}{\beta' X' X\beta}X'X\beta\right\}$$

$$-2\{u'W^{-1}X(X'W^{-1}X)^{-1}\}\left\{u'\beta u'(X'W^{-1}X)^{-1}XX'W^{-1}u\frac{2\beta' X'u}{\beta' X' X\beta}X'X\beta\right\}$$

$$+2\lambda_1\left\{\frac{T}{\alpha}X(X'W^{-1}X)^{-1}X'W^{-1}u'u + k(1-\lambda_2)\right..$$

$$\left\{u'BuX(X'W^{-1}X)^{-1}X'W^{-1}uu' - \frac{2\beta' X'u}{\beta' X' X\beta}u'X\beta\right\}$$

$$\left.-\frac{2T}{\alpha}\beta'\{(X'X)^{-1} - (X'W^{-1}X)^{-2}\}TX'uu'X\beta\right\}\Bigg]. \qquad (5.11)$$

Here we can see that

$$E[\phi_1] = \lambda_1(n-2p) + p. \qquad (5.12)$$

$$E[\phi_2] = -2k\gamma_1\frac{(1-\lambda_1)(1-\lambda_2)}{\beta' X' X\beta}\beta'(I_n * B)e. \qquad (5.13)$$

$$E[\phi_3] = (X'W^{-1}X)^{-1} - \frac{T}{\alpha}\beta' X' X\beta(X'W^{-1}X)^{-2} - (1-2\lambda_1)\frac{T}{\alpha}.$$

$$(X'W^{-1}X)^{-1} - 2k\gamma_1(1-\lambda_2)\frac{T}{\alpha}(X'W^{-1}X)^{-1}trB$$

$$-\gamma_2\left[\frac{k^2(1-\lambda_2)^2}{\beta' X' X\beta}trB(I_n * B)\right.$$

$$+2k(1-\lambda_1)(1-\lambda_2)(I_n * B) + 4k(1-\lambda_1)(1-\lambda_2)(I_n * B)]$$

$$\left.-\frac{k^2(1-\lambda_2)^2}{\beta' X' X\beta}\{\{trB.B + 2trB\} - 6k(1-\lambda_1)(1-\lambda_2)(trB + 2B)\}\right]. \qquad (5.14)$$

Utilizing (5.12), (5.13), (5.14) in (5.8) we get the result of (3.3) of the Theorem. Putting $\lambda_2 = 1$ in this result we obtain the result (3.4) and by substituting $\lambda_2 = 0$ we get the result (3.5).

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