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SOLUTION OF THE DIOPHANTINE EQUATION

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ABSTRACT :- The Diophantine equation

$$2 + x + x^3 + 2x^4 = y^2$$

has no integer solution

Keywords :-
Equation, Divisor, integer, Diophantine

Introduction :-

Diophantine equations are polynomial equations where all the variables are integers, and solving them for integer solutions can be quite complex. While some specific Diophantine equations have elementary proofs (meaning proofs that rely only on basic arithmetic), the field in general requires more advanced techniques. Mathematics thrives on equations. relationships between variables that unlock a universe of possibilities. Yet, some problems demand a more specific kind of solution: whole numbers, the integers that form the backbone of counting and discrete quantities. This is the domain of Diophantine equations, named after the Hellenistic mathematician Diophantus, and within this realm, linear Diophantine equations hold a place of fundamental importance. A linear Diophantine equation is a linear equation where the unknowns and the constants are all integers. We typically express them in the from ax + by = c, where a, b, and care integers, and x and y are the integer unknowns we seek. The beauty of linear Diophantine equations lies in their relative simplicity and the rich tapestry of applications they offer. Determining if solutions exist and finding them often hinges on the greatest common divisor (GCD) of a and b. If the GCD does not divide c, the equation has no integer solutions. However, if the GCD divides c, then there exist infinitely many integer solutions. This paves the way for elegant algorithms to find particular solutions and express all solutions in a general form.

There are three types of DiophantineEquations :-

a) The equation which have no solutions

b) The equation which have only infinitely many solutions.

c) The equation which have only finitely many solutions.

The above Diophantine Equation

$$2 + x + x^3 + 2x^4 = y^2$$

is of the type (a).Congruence methods provide a useful tool in determining the number of solution to this diophantine equation.

$$2 + x + x^3 + 2x^4 = y^2$$

Proof:-

The Diophantine equation

$$2 + x + x^3 + 2x^4 = y^2 \quad (1)$$

has no integer solutions

The equation (1) can be written as

$$2x^2 + x^3 - 2x^4 - 2 + x - 2x^2$$
$$x^2(2 + x - 2x^2) + 1(2 + x - 2x^2)$$
$$(x^2 + 1)(2 + x - 2x^2) = y^2$$

Let D be the g.c.d of two terms in left hand side. D divides

$$-2(x^2 + 1) - (2 + x - 2x^2) = -(x + 4)$$

and also D divides

$$(x^2 + 1) - x(x + 4) = 4x + 1$$

Therefore D divides 1 +4
-4 1 | =1+16=17

Now from the equation (1) we have

$$x^2 + 1 = 17^j m^2$$

and $2 + x - 2x^2 = 17^j n^2$ (2)

Where $j = 0 \; or \; 1 \; and \; (m, n) = 1, m > 0, n > 0$

If x were odd, then m is even and the first equation in (2) is impossible to mod4, since $x^2 \equiv 1(mod4), and \; m^2 \equiv 0(mod4)$ so

$2 \equiv 0(mod4)$ is impossible.

Hence x is even and m odd and n even

Case3. If $j = 0$. from equation (2) we have

$$x^2 + 1 = m^2 \quad (3)$$

$2 + x - 2x^2 = n^2$

Where x is even, m is odd and n is even.

From equation first in (3) we have

$m^2 - x^2 = 1$

Or, $(m + x)(m - x) = 1$

So $m + x = \pm 1$

And $m - x = \pm 1$

So $2x = 0$

Thus $x = 0$

Hence $m = 1$

And $n^2 = 2$

Imply $n = \pm\sqrt{2}$ which is irrational where

But we have taken n rational

So the equation in(3) is impossible.

Case4. For $j = 1$, we have from the equation (4)

$$x^2 + 1 = 17m^2$$
$$2 + x - 2x^2 = 17n^2$$

Where x is even, m is odd and n is even.
multiplyingthe first equation of (4) by 2 and then adding the second we have

$$x + 4 = 17(n^2 + 2m^2)$$
$$x = -4 + 17(n^2 + 2m^2) \qquad (5)$$

Substitutiong this value of x in first equation of (4) we get

$$\{-4 + 17(n^2 + 2m^2)\}^2 + 1 = 17m^2$$

or, $16 + 2(-4).17(n^2 + 2m^2) + 17^2(n^2 + 2m^2)^2 + 1 = 17m^2$
or, $17 + 2.(-4).17(n^2 + 2m^2) + 17^2(n^2 + 2m^2)^2 = 17m^2$
Dividing both sides by 17 we have

$$1 + 2.(-4)(n^2 + 2m^2) + 17(n^2 + 2m^2)^2 = m^2$$

or,$1 - 2(1).4(n^2 + 2m^2) + 16(n^2 + 2m^2)^2 + (n^2 + 2m^2)^2 = m^2$
or, $1 - 2.1.4(n^2 + 2m^2) + \{4(n^2 + 2m^2)\}^2 + (n^2 + 2m^2)^2 = m^2$
or, $\{1 - 4(n^2 + 2m^2)\}^2 + (n^2 + 2m^2)^2 = m^2$ (6)
Now $\{1 - 4(n^2 + 2m^2), n^2 + 2m^2\}$

$$= \{1 - 4(n^2 + 2m^2) + 4(n^2 + 2m^2), n^2 + 2m^2\}$$
$$= \{1, n^2 + 2m^2\}$$
$$= 1$$

So the numbers on the left hand sides in brackets in equation (6) are relatively prime and this equation (6) of the pythagorean type we have

$$1 - 4(n^2 + 2m^2) = \pm(r^2 - s^2)$$
$$n^2 + 2m^2 = 2rs$$
$$m = (r^2 + s^2) > 0 \qquad (7)$$

Where $(r, s) = 1$ and r, s are of opposite parity since m is odd and n is even.
Here $rs \neq 0$, Since if $rs = 0$then

$$n^2 + 2m^2 = 0$$

Which is impossible because

$$m > 0, n > 0$$

Now if $rs \neq 0$then from equations (7) we have

$$1 - 8rs = \pm(r^2 - s^2)$$
$$n^2 = -2(r^2 + s^2)^2 + 2rs$$

Where n even $(r, s) = 1$and r, s of opposite parity which is impossible because the square of an even integer

$$\equiv 0, 1 \; or \; 4 \; (mod 8)$$

Hence the theorem.

Conclusion

Elementary solutions play a crucial role in solving specific Diophantine equations. From the straightforward approach of linear equations to the more intricate methods for Pell's equation, these techniques showcase the power of manipulating integers and their properties. However, it's valuable to recognize the limitations of these methods and appreciate the vast landscape of Diophantine equations that may require more sophisticated approaches. As mathematicians continue to explore this fascinating realm, elementary solutions will remain a cornerstone for understanding and solving specific Diophantine equations.

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