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A Class of Integer solutions to Homogeneous Cubic Diophantine Equation with Four Unknowns

$$(x+y)\,[3(x-y)^2+xy]=20zp^2$$

Dr. N.Thiruniraiselvi[1*], Dr. M.A.Gopalan[2]

[1] Assistant Professor, Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Samayapuram, Trichy- 621 112, Tamil Nadu, India.
email:drntsmaths@gmail.com.

[2] Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India.
email:mayilgopalan@gmail.com

Abstract

The process of obtaining different choices of integer solutions to Homogeneous Cubic Diophantine Equation with four unknowns given by $(x+y)\,[3(x-y)^2+xy]=20zp^2$ has been illustrated. The substitution strategy andfactorization method are utilized to solve the above cubic equation.Keywords :Homogeneous cubic ,Cubic with four unknowns ,Integer solutions

Introduction

Diophantine equations , one of the areas of number theory ,occupy a pivotal role in the realm of mathematics and have a wealth of historical significance. It is well-known that Diophantine equations are rich in variety. Particularly ,finding integer solutions to homogeneous cubic equation with four unknowns is a topic of current research. While collecting problems on the same , the article presented in [1] was noticed and the authors have obtained three sets of integer solutions in a particular pattern..However ,there are many more fascinating patterns of solutions in integers . The main thrust of this paper is to exhibit other solution patterns to the homogeneous cubic equation with four unknowns

$(x+y)[3(x-y)^2+xy]=20zp^2$.

Method of analysis

The homogeneous cubic equation with four unknowns to be solved is

$$(x+y)\,[3(x-y)^2+xy]=20zp^2 \qquad (1)$$

The process of obtaining varieties of integer solutions to (1) are as follows:

Process 1

Taking

$$x=3u+\alpha+20\beta,\ y=3u-\alpha-20\beta, z=6u, p=\alpha+11\beta \qquad (2)$$

in (1) , it reduces to the quadratic equation

$$\alpha^2=220\beta^2+u^2 \qquad (3)$$

which is satisfied by

$$\beta=2rs, u=220r^2-s^2, \alpha=220r^2+s^2 \qquad (4)$$

In view of (2) , the integer solutions to (1) are given by

$$x=880r^2-2s^2+40rs,\ y=440r^2-4s^2-40rs,$$
$$z=1320r^2-6s^2, p=220r^2+s^2+22rs$$

Note 1

Apart from (4) , there are twenty more choices of solutions to (3) that are

presented in Table 1 below:

Table 1-values of β,α,u

β	α	u
2 k	221 k	-219 k ,219 k
k	56 k	-54 k ,54 k
2 k	59 k	-51 k ,51 k
2 k	49 k	-39 k ,39 k
k	16 k	-12 k,12 k
2 k	31 k	-9 k ,9k
2k	$2k^2+110$	$2k^2-110$
k	k^2+55	k^2-55
2k	$10k^2+22$	$10k^2-22$
k	$5k^2+11$	$5k^2-11$
2k	$22k^2+10$	$22k^2-10$
k	$11k^2+5$	$11k^2-5$
2k	$110k^2+2$	$110k^2-2$
k	$55k^2+1$	$55k^2-1$

Employing (2) ,the corresponding integer solutions to (1) are obtained.

Process 2

Write (3) as

$$u^2+220\beta^2=\alpha^2*1 \qquad (5)$$

Assume

$$\alpha=a^2+220b^2 \qquad (6)$$

Express the integer 1 on the R.H.S. of (5) as

$$1=\frac{(6+i\sqrt{220})\,(6-i\sqrt{220})}{256} \qquad (7)$$

Substituting (6) & (7) in (5) and employing the method of factorization,

we consider

$$u+i\sqrt{220}\,\beta=\frac{(6+i\sqrt{220})}{16}[a+i\sqrt{220}\,b]^2 \quad (8)$$

Equating the real and imaginary parts in (8) and replacing a by $4A$ &

b by $2B$, the corresponding integer solutions to (1) are given by

$$x=54A^2-1210B^2-540AB,\ y=-18A^2-770B^2-780AB,$$
$$z=6(6A^2-330B^2-220AB),\ p=27A^2+275B^2+66AB$$

Note 2

In addition to (7) ,one may have

$$1=\frac{(220r^2-s^2+i\sqrt{220}\,2rs)\,(220r^2-s^2-i\sqrt{220}\,2rs)}{(220r^2+s^2)^2},$$
$$1=\frac{(r^2-220s^2+i\sqrt{220}\,2rs)\,(r^2-220s^2-i\sqrt{220}\,2rs)}{(r^2+220s^2)^2}$$

Following the above procedure , two more sets of integer solutions

to (1) are obtained.

Process 3

Taking

$$x=u+v,\ y=u-v,\ z=2u,\ u\neq v\neq 0 \qquad (9)$$

in (1) , it reduces to the quadratic equation

$$u^2+11v^2=20p^2=20p^2*1 \quad (10)$$

Assume

$$p=a^2+11b^2 \quad (11)$$

Consider

$$20=(3+i\sqrt{11})(3-i\sqrt{11}) \quad (12)$$

Express the integer 1 on the R.H.S. of (10) as

$$1=\frac{(5+i\sqrt{11})(5-i\sqrt{11})}{36} \quad (13)$$

Following the procedure as in Process 2 , the corresponding integer solutions

to (1) are found to be

$$x=18(A^2-11B^2)-252AB,\ y=-6(A^2-11B^2)-276AB,$$
$$z=12(A^2-11B^2)-528AB,\ p=9(A^2+11B^2)$$

Note 3

In addition to (13) ,one may have

$$1=\frac{(11r^2-s^2+i\sqrt{11}\,2rs)(11r^2-s^2-i\sqrt{11}\,2rs)}{(11r^2+s^2)^2},$$
$$1=\frac{(r^2-11s^2+i\sqrt{11}\,2rs)(r^2-11s^2-i\sqrt{11}\,2rs)}{(r^2+11s^2)^2},$$
$$1=\frac{(1+i3\sqrt{11})(1-i3\sqrt{11})}{100},$$
$$1=\frac{(7+i5\sqrt{11})(7-i5\sqrt{11})}{324}.$$

Following the above procedure , four more sets of integer solutions to (1) are obtained.

Process 4

Rewrite (10) in the form of ratios as

$$\frac{u+3p}{p+v}=\frac{11(p-v)}{u-3p}=\frac{A}{B}\ ,B\neq 0\,(14)$$

Solving (14) as the system of double equations through the method of cross-multiplication , the corresponding integer solutions to (1) are given by

$$x=2\,(\,A^2-11B^2)+28AB\ ,y=4\,(\,A^2-11B^2)+16AB\ ,$$
$$z=6\,(\,A^2-11B^2)+44AB\ \ ,p=(\,A^2+11\,B^2)$$

Reference

[1] G. Janaki , S.Sarumathi ,Holistic Solutions of Cubic Diophantine Equation with Four unknowns ,IJRASET,Vol.12 ,Issue III ,1038-1043,2024