# On the principle of exchange of stabilities in magnetohydrodynamic thermohaline convection

Dr. RajBala
*Govt. PG College, Sector – 1, Panchkula – 134 109, Haryana, India*
*E-mail*: rajbala07@gmail.com

**Abstract**

In the present paper, we have discussed the case of principle of exchange of stabilities for generalized magnetohydrodynamic Bernard convection problem in the case when $\tau \neq 0$ and derived a new modified Rayleigh number denoted by $\Re = [R_1(1-\alpha_2 T_0) - \frac{R_2}{\tau} + T_0\hat{\alpha}_2 R_1 R_3]$ and an extended modified Rayleigh number $\Re' = [R_1(1-\alpha_2 T_0) - \frac{R_2}{\tau} + T_0\hat{\alpha}_2 R_1 R_3 - Q\Pi^2]$. We further derived conditions for the principle of exchange of stabilities and overstability respectively. Simple and different modified Bernard convection problems are obtained as special cases. The result for the case of overstability is also discussed through the variational principle.

***Keywords:*** Rayleigh number, Modified Rayleigh number, Extended modified Rayleigh number, Variational principle, Principle of exchange of stabilities, Overstability.

## 1. Introduction

Convection is the concerted collective movement of groups or aggregate of molecules within fluids either through advection or through diffusion or as a combination of both of them. Convection occurs on a large scale in atmospheres, oceans, planetary mantles and it provides the mechanism of heat transfer for a large fraction of the outermost interior of Sun and stars. Convection in a stably stratified fluid induced by the different molecular diffusivity of two components is known as 'double-diffusive convection'. The term 'double-diffusive convection' applies to the convection in a fluid where there are two diffusive constituents having effect on buoyancy. The physics behind double diffusive convection is quiet similar to the stellar case in which helium acts like salt in raising the density and making the diffusion slower than heat. When one of the diffusive component is heat, then double diffusion convection is called a thermohaline convection. Since, heat and salt diffuse through liquid at different rates in thermohaline convection, that is why it is also called as the 'double-diffusive convection'. Thermohaline convection problem or more generally double-diffusive convection problem has been of great interest in the recent past years due to its direct relevance to many practical problem in the fields of oceanography, astrophysics and chemical engineering etc. Examples of particular interest are provided by ponds build to trap solar heat ( Tabor and Matz 1965 [1]) and some Antartic lakes( Shirtcliffe 1964 [2]).

Two fundemental configuration in the context of the thermohaline instability problem have been studied, one by Stern [3], wherein the temperature gradient is stablizing and the concentration gradient is destablizing and another by Veronis [4], wherein the temperature gradient is destablizing and the concentration gradient is stablizing. The problem of thermohaline convection in a layer of fluid heated from below and subjected to stable salinity gradient has been first investigated by Veronis [4]. Later Banerjee [5] investigated thermal instability problem of non-homogeneous fluids. A detailed study on thermal and thermohaline convection has been done by Katoch [6]. A modified analysis of thermal/thermohaline instability of a liquid layer heated underside(Bernard convection) and under the effect of rotation has been discussed in past by Banerjee et. al. [7,8]. Gupta et.al. [9], Dhiman and Sharma [10] have successfully proved some important results on thermohaline convection problem of Stern and Veronis types in the recent years. A detailed account of thermal convection in a horizontal thin layer of Newtonian fluid heated from below, under varying assumption of hydrodynamics and hydromagnetics has been nicely described by Chandrashekhar [11]. Sharma and Rani [12], Sharma and Kumar [13], Rani and Tomar [14,15] including others extended thermohaline convection problems to micropolar fluids.

The transition from stability to instability occurs via a marginal stationary state. This state is characterized by the vanishing of both the real and imaginary parts of the complex time eigenvalue associated with the disturbance. If at the onset of instability, a stationary pattern of motions prevails i.e. the instablity sets in as stationary cellular convection then one says principle of exchange of stabilities is valid and if at the onset of instability, oscillatory motions prevail, then one says that one has a case of overstabilies. The principle of exchange of stabilities was used by Rayleigh [16] in the the first theoretical investigation of the thermal stability problem for a layer of fluid bounded by two infinite horizontal planes. Pellew and Southwell [17] proved the validity of principle of exchange of stabilities for the classical Rayleigh Bernard convection problem. Chandrashekhar [11] extended the proof of exchange principle developed by Pellew and Southwell [17] for the thermal stability problem to include the effect of fluid electrical conductivity and a uniform vertical magnetic field. Chandrashekhar [11] in his investigation of the magnetohydrodynamic simple Bernard convection problem sought unsuccessfully the regime in terms of parameters of the system alone, in which the total kinetic energy associated with a disturbance exceeds the total magnetic energy associated with it since these consideration are of decisive significance in deciding the validity of principle of exchange of stabilities. Banerjee et.al. [18,19] have discussed principle of exchange of stabilities in the case of magnetohydrodynamic simple Bernard problem and effect of magnetic field in hydromagnetic generalized Bernard problem . Later, Gupta and Rana [20] and others did work in magnetohydrodynamic for thermal and thermohaline problems.

Gupta and Rana [20] derived that principle of exchange of stabilities is not valid for generalized magnetohydrodynamic Bernard convection for the case $\tau = 0$ . In the present paper, we discussed the case when $\tau \neq 0$ and derived a modified Rayleigh number given by $\Re = [R_1(1-\alpha_2 T_0) - \frac{R_2}{\tau} + T_0\hat{\alpha}_2 R_1 R_3]$ and an extended modified Rayleigh number denoted by $\Re' = [R_1(1-\alpha_2 T_0) - \frac{R_2}{\tau} + T_0\hat{\alpha}_2 R_1 R_3 - Q\Pi^2]$ and concluded that the principle of exchange of stabilities is valid only if $\Re' > 0$ or if $R_1 > \dfrac{Q\Pi^2 + \frac{R_2}{\tau}}{1 + T_0(\hat{\alpha}_2 R_3 - \alpha_2)}$. Further the sufficient condition of overstability case has been also derived. Simple Bernard convection, modified simple Bernard convection, thermohaline convection of Veronis type, modified thermohaline convection of Veronis type, modified simple magnetohydrodynamic Bernard convection have been discussed as special cases. We also derived the critical Rayleigh numbers for different boundaries in the absence of magnetic field. For the case $Q = 0, \alpha_2 = \hat{\alpha}_2 = 0$, the equation reduces to the form as derived by Dhiman and Sharma [10], but in our case effective Rayleigh number is replaced by the modified Rayleigh number.The case of overstability is also verified through the variational principle.

## 2. Mathematical Formulation

Following Banerjee et al. [7,8], the modified simplified equations governing Bernard convection ($S = 0 = \Delta\rho'$), generalized Bernard convection ($\tau_0 = 0$), and themohaline convection of Veronis' type under the effect of a uniform vertical magnetic field are given by

$$\frac{\partial U_j}{\partial x_j} = 0, \tag{2.1}$$

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} - \frac{\mu_e}{4\pi\rho_0}H_j\frac{\partial H_i}{\partial x_j}$$

$$= -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0} + \frac{\mu_e|H|^2}{8\pi\rho_0}\right) + \left(1 + \frac{\Delta\rho}{\rho_0} + \frac{\Delta\rho'}{\rho_0}\right)X_i + \nu_0\nabla^2 U_i, \tag{2.2}$$

$$(1-\alpha_2 T)\left(\frac{\partial T}{\partial t}+U_j\frac{\partial T}{\partial x_j}\right)+\hat{\alpha}_2 T\left(\frac{\partial S}{\partial t}+U_j\frac{\partial S}{\partial x_j}\right)=\kappa_0\nabla^2 T, \tag{2.3}$$

$$\frac{\partial S}{\partial t}+U_j\frac{\partial S}{\partial x_j}=\tau_0\nabla^2 S, \tag{2.4}$$

$$\frac{\partial H}{\partial t}+U_j\frac{\partial H_i}{\partial x_j}=H_j\frac{\partial U_i}{\partial x_j}+\eta_0\nabla^2 H_i, \tag{2.5}$$

$$\frac{\partial H_j}{\partial x_j}=0, \tag{2.6}$$

and

$$\rho=\rho_0[1+\alpha(T_0-T)-\hat{\alpha}_2(S_0-S)], \tag{2.7}$$

where $i = 1,2,3, x_i's, u_i's, H_i's$ and $X_i's$ are respectively the cartesian coordinates x, y, z, velocity components u, v, w, magnetic field components $H_1, H_2, H_3$ and external force components $0, 0, -g; t$ is the time, p is the pressure; $\Delta\rho$ and $\Delta\rho'$ are respectively the variations in the density due to temperature and concentration; $\rho_0$, $\nu_0$, $\kappa_0$, $\tau_0$ and $\eta_0$ are respectively the values of density, viscosity, thermal diffusivity at the lower boundary; $\alpha_2$ and $\hat{\alpha}_2$ are respectively the coefficients of specific heat variation due to temperature and concentration variations; $\alpha$ and $\alpha'$ are respectively the coefficients of volume expansion due to temperature and concentration; T is the temperature; and S is the concentration.

## 3. Mathematical Analysis

Using the normal mode method and following the usual steps of linear stability theory, equations (2.1) to (2.7) rduced to the following non dimensional perturbation equations:

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w=R_1a^2\theta-\frac{R_2}{R_3}a^2\phi-QD(D^2-a^2)h_z, \tag{3.1}$$

$$[(D^2-a^2)-p(1-\alpha_2T_0)]\theta-T_0\hat{\alpha}_2p\phi=-(1-\alpha_2T_0)w-T_0\hat{\alpha}_2R_3w, \tag{3.2}$$

$$[\tau(D^2-a^2)-p]\phi=-R_3w, \tag{3.3}$$

$$(D^2-a^2-\frac{p\sigma_1}{\sigma})h_z=-Dw, \tag{3.4}$$

with the following boundary conditions:

$$\left.\begin{aligned}
&w = 0 = D^2w = D\theta = D\phi = h_z \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2},\\
&\text{or} \quad w = 0 = Dw = D\theta = D\phi = h_z \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2},\\
&\text{or} \quad w = 0 = D^2w = D\theta = D\phi \quad \text{and} \quad Dh_z = \pm az \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2},\\
&\text{or} \quad w = 0 = Dw = D\theta = D\phi \quad \text{and} \quad Dh_z = \pm ah_z \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2},\\
&\text{or} \quad w = 0 = D^2w = \theta = \phi = h_z \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2},\\
&\text{or} \quad w = 0 = Dw = \theta = \phi = h_z \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2},\\
&\text{or} \quad w = 0 = D^2w = \theta = \phi \quad \textit{and} \quad Dh_z = \pm ah_z \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2},\\
&\text{or} \quad w = 0 = Dw = \theta = \phi \quad \textit{and} \quad Dh_z = \pm ah_z \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2}.
\end{aligned}\right\} \tag{3.5}$$

In the above equations, z is the vertical coordinate, $z = -\frac{1}{2}$ and $z = +\frac{1}{2}$ represent the two boundaries, $D = d/dz$, $w \to$ vertical velocity, $a^2 \to$ square of the wave number, $h_z \to$ vertical magnetic field, $\theta \to$ temperature, $\phi \to$ concentration, $\sigma \to$ thermal Prandtl number, $\sigma_1 \to$ magnetic Prandtl number, $Q \to$ Chandrashekhar number, $(R_1 > 0) \to$ Rayleigh number, $(R_2 > 0) \to$ concentration Rayleigh number, $R_3 \to$ ratio of concentration gradient to temperature gradient, $\tau \to$ ratio of mass diffusivity to heat diffusivity, $\alpha_2$ & $\hat{\alpha}_2$ are respectively the coefficients of specific variation due to temperature and concentration variations, $p = p_r + ip_i$ is the complex growth rate.

**Theorem 1**: If $(p, w, \theta, \phi, h_z)$ is a solution of equations (3.1)-(3.4) together with either of the boundary conditions (3.5) and $\tau \neq 0$, then principle of exchange of stabilities is valid if

$$R_1 > \frac{(Q\Pi^2 + \frac{R_2}{\tau})}{1 + T_0(\hat{\alpha}_2 R_3 - \alpha_2)}. \tag{3.6}$$

**Proof**: For $\tau \neq 0$, let if possible principle of exchange of stabilities is valid i.e. $p = 0$, then equations (3.1)- (3.4) reduce to

$$(D^2 - a^2)^2 w = R_1 a^2 \theta - \frac{R_2}{R_3} a^2 \phi - QD(D^2 - a^2)h_z, \tag{3.7}$$

$$(D^2 - a^2)\theta = -(1 - \alpha_2 T_0)w - T_0 \hat{\alpha}_2 R_3 w, \tag{3.8}$$

$$(D^2 - a^2)\phi = -\frac{R_3}{\tau} w, \tag{3.9}$$

and

$$(D^2 - a^2)h_z = -Dw, \tag{3.10}$$

together with boundary conditions given by (3.5).
Now, using the transformation $\psi = R_1\theta - \frac{R_2}{R_3}\phi$ , equations (3.7)-(3.10) reduce to

$$(D^2 - a^2)^2 w = a^2\psi + QD^2 w, \tag{3.11}$$

$$(D^2 - a^2)\psi = -[R_1(1 - \alpha_2 T_0) - \frac{R_2}{\tau} + T_0\hat{\alpha}_2 R_1 R_3]w, \tag{3.12}$$

$$(D^2 - a^2)\psi = -\Re w, \tag{3.13}$$

where

$$\Re = [R_1(1 - \alpha_2 T_0) - \frac{R_2}{\tau} + T_0\hat{\alpha}_2 R_1 R_3], \tag{3.14}$$

is a **modified thermohaline Rayleigh number**.
Now, multiplying equation (3.11) by $w^*$ and integrating the equations so obtained over the vertical range of z by parts a finite number of times and using the boundary conditions (2.5), we get

$$\int_{-1/2}^{1/2} w^*(D^2 - a^2)^2 w dz = a^2 \int_{-1/2}^{1/2} w^*\psi dz + Q\int_{-1/2}^{1/2} w^* D^2 w dz. \tag{3.15}$$

$$\Rightarrow \int_{-1/2}^{1/2} [|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2]dz = a^2 \int_{-1/2}^{1/2} w^*\psi dz - Q\int_{-1/2}^{1/2} |Dw|^2 dz, \tag{3.16}$$

from (3.13), we can write

$$\int_{-1/2}^{1/2} \psi(D^2 - a^2)\psi^* dz = -\Re \int_{-1/2}^{1/2} \psi w^* dz, \tag{3.17}$$

or

$$\frac{1}{\Re}\int_{-1/2}^{1/2} [|D\psi|^2 + a^2|\psi|^2]dz = \int_{-1/2}^{1/2} w^*\psi dz. \tag{3.18}$$

where using (3.18) in (3.16), one gets

$$\int_{-1/2}^{1/2} [|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2]dz = \frac{a^2}{\Re}\int_{-1/2}^{1/2} [|D\psi|^2 + a^2|\psi|^2]dz - Q\int_{-1/2}^{1/2} |Dw|^2 dz, \tag{3.19}$$

or

$$\int_{-1/2}^{1/2} |D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2]dz + Q\int_{-1/2}^{1/2} |Dw|^2 - \frac{a^2}{\Re}\int_{-1/2}^{1/2} [|D\psi|^2 + a^2|\psi|^2]dz = 0. \tag{3.20}$$

Again from (3.13), we have

$$\int_{-1/2}^{1/2}(D^2-a^2)\psi(D^2-a^2)\psi^* dz = \Re^2\int_{-1/2}^{1/2}|w|^2dz, \tag{3.21}$$

$$\int_{-1/2}^{1/2}[|D^2\psi|^2+2a^2|D\psi|^2+a^4|\psi|^2]dz = \Re^2\int_{-1/2}^{1/2}|w|^2dz, \tag{3.22}$$

$$\int_{-1/2}^{1/2}[|D\psi|^2+a^2|\psi|^2 \leq \frac{\Re^2}{a^2}\int_{-1/2}^{1/2}|w|^2dz, \tag{3.23}$$

$$\leq \frac{\Re^2}{a^2\Pi^2}\int_{-1/2}^{1/2}|Dw|^2dz, \tag{3.24}$$

where (3.24) is obtained by using Rayleigh Ritz inequality.

Now using (3.24) in (3.20), we get

$$\int_{-1/2}^{1/2}[|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]dz + [Q-\frac{\Re}{\Pi^2}]\int_{-1/2}^{1/2}|Dw|^2dz \leq 0. \tag{3.25}$$

It is clear that equation (3.25) is not possible until

$$Q\Pi^2 < \Re, \tag{3.26}$$

or

$$Q\Pi^2 < [1-\alpha_2T_0+T_0\hat{\alpha_2}R_3]R_1 - \frac{R_2}{\tau}, \tag{3.27}$$

which further implies that

$$(1-\alpha_2T_0+T_0\hat{\alpha_2}R_3)R_1 - \frac{R_2}{\tau} - Q\Pi^2 > 0 \quad \text{or} \quad \Re' > 0, \tag{3.28}$$

where

$$\Re' = (1-\alpha_2T_0+T_0\hat{\alpha_2}R_3)R_1 - \frac{R_2}{\tau} - Q\Pi^2, \tag{3.29}$$

is our new **extended modified Rayleigh number**.

Using equation (3.28), we can say that the principle of exchange of stabilities is valid only if

$$\Re' > 0 \quad \text{or} \quad R_1 > \frac{Q\Pi^2+\frac{R_2}{\tau}}{[1-T_0(\alpha_2-\hat{\alpha_2}R_3)]}, \tag{3.30}$$

and therefore, we can conclude that principle of exchange of stabilities is not valid if

$$R_1 \leq \frac{Q\Pi^2+\frac{R_2}{\tau}}{[1-T_0(\alpha_2-\hat{\alpha_2}R_3)]}, \tag{3.31}$$

hence we get the critical Rayleigh number.
From the results obtained above, we also get a sufficient condition for the validity of overstability, which says that

$$R_1 \leq \frac{Q\Pi^2 + \dfrac{R_2}{\tau}}{[1 - T_0(\alpha_2 - \hat{\alpha}_2 R_3)]}. \tag{3.32}$$

From (3.30), we derive the following results :
(i) As $R_2 > 0$ increases and other parameters are given fixed values, the value of $R_1$ increases. It means that in a Veronis type instability problem, the role of concentration gradient is to stabilize the flow.
(ii) As Q increases and other parameters are kept constant, $R_1$ increases. So, we can conclude that the role of magnetic field is to stablize the fluid layer.
(iii) For given $Q > 0,\ R_2 > 0,\ \tau > 0$, as the temperature $T_0$ increases, $R_1$ decreases provided

$$R_3 > \frac{\alpha_2}{\hat{\alpha}_2}, \tag{3.33}$$

which indicates that hotter the liquid layer, the instability is preponed.
(iv) For given $Q > 0,\ R_2 > 0,\ \tau > 0$, as the temperature $T_0$ increases, $R_1$ increases provided

$$R_3 < \frac{\alpha_2}{\hat{\alpha}_2}, \tag{3.34}$$

which signifies that hotter the liquid layer, the instability is postponed.

## 4. Special cases

**(1) Modified thermohaline convection of Veronis type**: In the absence of magnetic field i.e. when Q=0, then from(3.29), we have

$$\Re' = (1 - \alpha_2 T_0 + T_0 \hat{\alpha}_2 R_3) R_1 - \frac{R_2}{\tau} = \Re, \tag{4.1}$$

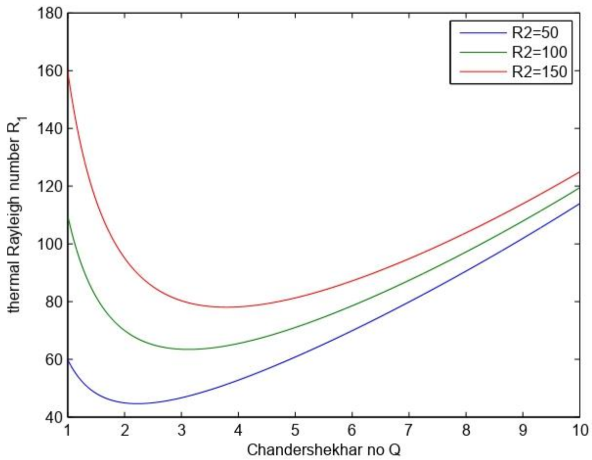


Figure 1: Plot of $R_1$ versus Q for different values of $R_2$

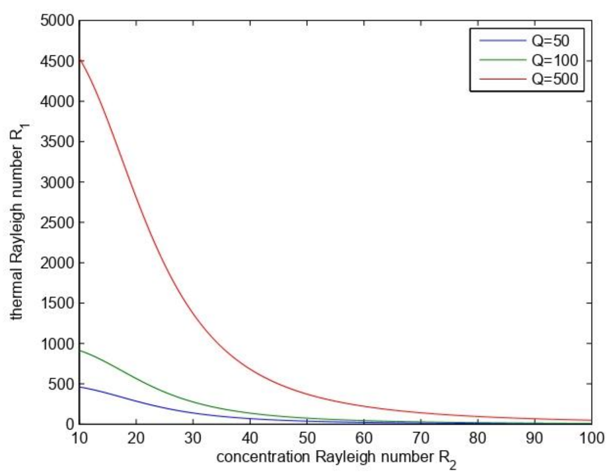


Figure 2: Plot of $R_1$ versus $R_2$ for different values of Q

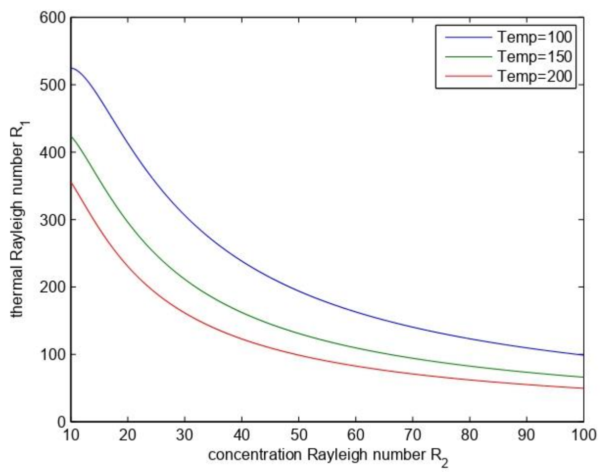


Figure 3: Plot of $R_1$ versus $R_2$ for different values of $T_0$(temperature)

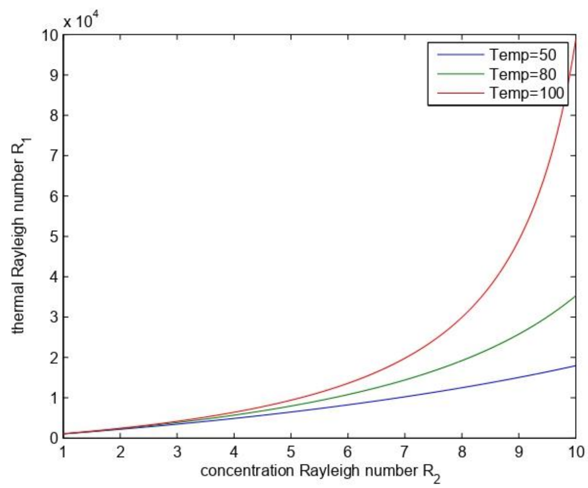


Figure 4: Plot of $R_1$ versus $R_2$ for different values of $T_0$(temperature)

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which is our modified Rayleigh number. Hence, from(4.1), we can say that PES is valid only if

$$R_1 > \frac{\frac{R_2}{\tau}}{[1 - T_0(\alpha_2 - \hat{\alpha}_2 R_3)]}, \tag{4.2}$$

which gives the critical Rayleigh number in the case of modified thermohaline convection of Veronis type.

**(2) Thermohaline convection of Veronis type**:

When $\alpha_2 = 0 = \hat{\alpha}_2 = Q$, then

$$\Re' = R_1 - \frac{R_2}{\tau}, \tag{4.3}$$

which coincides with the effective Rayleigh number obtained by Dhiman and Sharma [20]. Hence, PES is valid if

$$R_1 > \frac{R_2}{\tau}, \tag{4.4}$$

which gives the critical Rayleigh number in case of thermohaline convection of Veronis type.

**(3) Modified simple magnetohydrodynamic Bernard convection**:

In the absence of solute concentration i.e. when $\hat{\alpha}_2 = 0 = R_2$, then

$$\Re' = (1 - \alpha_2 T_0)R_1 - Q\Pi^2. \tag{4.5}$$

Hence, PES is valid if

$$R_1 > \frac{Q\Pi^2}{(1 - \alpha_2 T_0)}, \tag{4.6}$$

which gives the critical Rayleigh number in the case of modified simple magnetohydrodynamic Bernard convection.

**(4) Modified simple Bernard convection**:

When $\hat{\alpha}_2 = 0 = R_2 = Q$, then

$$\Re' = (1 - \alpha_2 T_0)R_1. \tag{4.7}$$

**(5) Simple Bernard convection**:

When $\alpha_2 = \hat{\alpha}_2 = 0 = R_2 = Q$, then

$$\Re' = R_1, \tag{4.8}$$

which is the classical Rayleigh number in simple Bernard convection problem (see Chanderasekhar, pp. 43 [4]).

## 5. Critical Rayleigh numbers in the absence of magnetic field

In the absence of magnetic field i.e. when $Q = 0$, the equations (3.11) and (3.13) reduce to

$$(D^2 - a^2)^2 w = a^2 \psi, \tag{5.1}$$

$$(D^2 - a^2)\psi = -\Re w, \tag{5.2}$$

where

$$\Re = [R_1(1 - \alpha_2 T_0) - \frac{R_2}{\tau} + T_0 \hat{\alpha_2} R_1 R_3], \tag{5.3}$$

with boundary conditions

$$\psi = 0 \quad \text{at} \quad z = \pm\frac{1}{2} \quad \text{with either} \quad Dw = 0 \quad \text{or} \quad D^2 w = 0 \quad \text{at} \quad z = \pm\frac{1}{2}. \tag{5.4}$$

Now equation (5.1) and (5.2) are identical to the classical Bernard equation, where $\Re$ plays the same role as R in the Rayleigh Bernard convection problem. Therefore, all the results which were valid for those equations also hold good for equations (5.1) and (5.2) for the modified thermohaline convection problem of Veronis type.

So, following the analysis of Chandershekhar [4], equations (5.1) to (5.2) with the relevant boundary conditions (5.4), yield the values of critical Rayleigh numbers as

**Case (1)**: *When both the boundaries are dynamically free*, then

$$\Re_c = \frac{27\Pi^4}{4} = 657.51. \tag{5.5}$$

Using(5.3), (5.5) further implies

$$R_{1_c}[1 - T_0(\alpha_2 - \hat{\alpha_2} R_3)] - \frac{R_2}{\tau} = 657.51, \tag{5.6}$$

or

$$R_{1_c} = \frac{657.51 + \frac{R_2}{\tau}}{[1 - T_0(\alpha_2 - \hat{\alpha_2} R_3)]}. \tag{5.7}$$

**Case (2)**: *When one boundary is rigid and one is free.*

$$R_{1_c} = \frac{1100.65 + \frac{R_2}{\tau}}{[1 - T_0(\alpha_2 - \hat{\alpha_2} R_3)]}. \tag{5.8}$$

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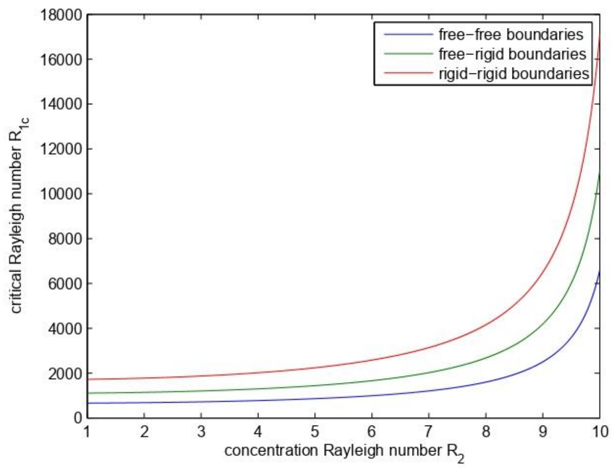


Figure 5: Plot of critical Rayleigh numbers versus $R_2$ for different boundaries

**Case (3)**: *When both the boundaries are rigid.*

$$R_{1_c} = \frac{1707.76 + \dfrac{R_2}{\tau}}{[1 - T_0(\alpha_2 - \hat{\alpha_2} R_3)]}. \tag{5.9}$$

where the values 657.51, 1100.65, 1707.76 are respectively the values of the critical thermal Rayleigh numbers for free-free, rigid-free, rigid-rigid boundary conditions for the simple Bernard convection problem (see Chanderasekhar, pp 43 [1]). The results for different boundaries are shown in Figure 5 and the values of critical Rayleigh numbers for different values of concentration Rayleigh number $R_2$ for the free-free, rigid-free, rigid-rigid boundaries are shown in Table 1.

## 6. Variational Principle

From equations (5.11) and (5.13), we can write

$$(D^2 - a^2)[(D^2 - a^2)^2 w - QD^2 w] = -a^2 \Re w, \tag{6.1}$$

Table 1: Critical Rayleigh numbers for different boundaries

| $R_2$ | free-free | rigid-free | rigid-rigid |
|---|---|---|---|
| 1 | 665.1 | 1113 | 1726 |
| 2 | 685.4 | 1147 | 1778 |
| 3 | 721.5 | 1207 | 1872 |
| 4 | 778.0 | 1302 | 2019 |
| 5 | 863.6 | 1445 | 2241 |
| 6 | 995.3 | 1665 | 2583 |
| 7 | 1210 | 2024 | 3139 |
| 8 | 1601 | 2679 | 4156 |
| 9 | 2505 | 4190 | 6500 |
| 10 | 6585 | 11020 | 17090 |

where $\Re$ is our modified Rayleigh number.
Letting

$$F = (D^2 - a^2)^2 w - QD^2 w, \tag{6.2}$$

Rewriting equation (6.1), we get

$$(D^2 - a^2)F = -\Re a^2 w, \tag{6.3}$$

with boundary conditions

$$F = 0, \quad w = 0, \quad D^2 w = 0 \quad \text{for} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2}. \tag{6.4}$$

Multiplying equation (6.3) by F and integrating over the range of z, we get

$$\int_{-1/2}^{1/2} F(D^2 - a^2)F dz = -\Re a^2 \int_{-1/2}^{1/2} wF dz, \tag{6.5}$$

$$\int_{-1/2}^{1/2} [|DF|^2 + a^2|F|^2] dz = \Re a^2 \int_{-1/2}^{1/2} wF dz, \tag{6.6}$$

$$= \Re a^2 \int_{-1/2}^{1/2} w[(D^2 - a^2)^2 w - QD^2 w] dz, \tag{6.7}$$

$$= \Re a^2 \int_{-1/2}^{1/2} [|(D^2 - a^2)w|^2 + Q|Dw|^2] dz, \tag{6.8}$$

$$\Rightarrow \Re = \frac{\int_{-1/2}^{1/2} [|DF|^2 + a^2|F|^2] dz}{a^2 \int_{-1/2}^{1/2} [|(D^2 - a^2)w|^2 + Q|Dw|^2] dz} = \frac{I_1}{a^2 I_2} > 0. \tag{6.9}$$

Equation (6.9) expresses $\Re$ as the ratio of two positive integrals and gives the critical Rayleigh number for the onset of instability as stationary convection which is the absolute minimum of the quantity on RHS of equation (6.9).
From equation (6.9), $\Re > 0$ implies that

$$R_1 > \frac{\frac{R_2}{\tau}}{[1 - T_0(\alpha_2 - \hat{\alpha}_2 R_3)]}, \tag{6.10}$$

so for overstability

$$R_1 \leq \frac{\frac{R_2}{\tau}}{[1 - T_0(\alpha_2 - \hat{\alpha}_2 R_3)]}. \tag{6.11}$$

Also for the case of overstability

$$R_1 \leq \frac{Q\Pi^2}{[1 - T_0(\alpha_2 - \hat{\alpha}_2 R_3)]}, \tag{6.12}$$

a result obtained by Gupta and Rana [15]. Hence, from (6.11) and (6.12), we can say that the condition for overstability is

$$R_1 \leq \frac{Q\Pi^2 + \frac{R_2}{\tau}}{[1 - T_0(\alpha_2 - \hat{\alpha}_2 R_3)]}, \tag{6.13}$$

which proves the validity of result obtained in Theorem 1.

## 7. Numerical results and conclusion

In order to verify our theoretical results, computational work has been carried out to get better clairity of the results obtained in the present paper. The effect of various parameters present in the system has been produced graphically which gives a fair and clear idea of their impact on the stability and instability of the system. The results obtained theoretically, graphically and numerically have been discussed as below:
In **Figure 1**, the graph has been drawn for thermal Rayleigh number $R_1$ versus Q (Chandershekhar number) by giving the values to different parameters as $\alpha_2 = (10^{-3}, 10^{-2})$, $\hat{\alpha}_2 = (10^{-4}, 10^{-3})$, $\tau = (10^0, 10^1)$, $R_3 = (10^{-1}, 10^0)$, $Q = (10^0, 10^1)$, $T_0 = (10^0, 10^1)$ except for the concentration Rayleigh number $R_2$ which is taking fixed values 50, 100, and 150. It is clear from the graph that as the value of concentration Rayleigh number $R_2$ is increasing, the value of $R_1$ is also increasing, thus making it clear that in a Veronis type instability problem, the role of concentration gradient is to stablize the fluid layer.
In **Figure 2**, the graph has been drawn for thermal Rayleigh number $R_1$ versus concentration Rayleigh number $R_2$ by giving the values to different parameters as $\alpha_2 = (10^{-3}, 10^{-2})$,

$\hat{\alpha}_2 = (10^{-2}, 10^{-1})$, $\tau = (10^0, 10^1)$, $R_3 = (10^0, 10^1)$, $R_2 = (10^1, 10^2)$, $T_0 = (10^1, 10^2)$ and Q(Chandershekhar number)=( 50, 100 and 500). Here the graph shows that as the effect of magnetic field is increasing, the value of $R_1$ is also increasing, thus making it clear that the effect of magnetic field is to stablize the fluid layer. Further for $R_2$ lying between 10 to 40, the difference in the values of $R_1$ is quite large and when $R_2 > 40$, the difference is gradually decreasing and the graphs almost coincide.

In **Figure 3** and **Figure 4**, the graphs have been drawn for thermal Rayleigh number $R_1$ versus concentration Rayleigh number $R_2$. In **Figure 3**, the values to different parameters have been taken as $\alpha_2 = (10^{-3}, 10^{-2})$, $\hat{\alpha}_2 = (10^{-2}, 10^{-1})$, $\tau = (10^0, 10^1)$, $R_3 = (10^0, 10^1)$, $Q = (10^2, 10^3)$, $R_2 = (10^0, 10^1)$ except $T_0$ (temperature) which is taking fixed values 100, 150 and 200. It is clear from the graph that as the temperature is increasing, the value of $R_1$ is decreasing. Which depicts that the effect of temperature is to prepone the instability when $R_3 > \frac{\alpha_2}{\hat{\alpha}_2}$.

In **Figure 4** the values to different parameters have been taken as $\alpha_2 = (10^{-3}, 10^{-2})$, $\hat{\alpha}_2 = (10^{-4}, 10^{-3})$, $\tau = (10^0, 10^1)$, $R_3 = (10^{-1}, 10^0)$, $Q = (10^2, 10^3)$, $R_2 = (10^0, 10^1)$ and $T_0 = (50, 80 \text{ and } 100)$. So from the graph, we can conclude that the role of temperature is reversed when $R_3 < \frac{\alpha_2}{\hat{\alpha}_2}$, that means in this case, temperature is making the fluid layer stable and postpones the onset of instability.

In **Figure:5**, the graph has been plottted for critical Rayleigh number $R_{1_c}$ versus concentration Rayleigh number $R_2$ by giving the values to different parameters as $\alpha_2 = (10^{-3}, 10^{-2})$, $\hat{\alpha}_2 = (10^{-4}, 10^{-3})$, $\tau = (10^0, 10^1)$, $R_3 = (10^{-1}, 10^0)$, $R_2 = (10^0, 10^1)$, $T_0 = (10^1, 10^2)$ for different boundaries.The theoretical results, graph and table are showing the same tendencies for free-free, free-rigid and rigid-rigid boundaries. The values of critical thermal Rayleigh numbers $R_{1_c}$ are increasing as the concentration Rayleigh number $R_2$ is varying from 1 to 10. The values of critical Rayleigh numbers are increasing gradually for free-free, free-rigid and rigid-rigid boundaries respectively.

Finally, we draw the following conclusions from this study:-

(i) We derived the condition of validity of principle of exchange of stability and found that the result holds if extended modified Rayleigh number

$$\Re' > 0 \quad \text{or} \quad R_1 > \frac{Q\Pi^2 + \frac{R_2}{\tau}}{1 + T_0(\hat{\alpha}_2 R_3 - \alpha_2)}.$$

(ii) In a Veronis type instability problem, the role of concentration gradient is to stabilize the flow by postponing the instability.

(iii) The role of magnetic field is to make the system stable by delaying instability in the system.

(iv) The role of temperature is to destabilize the fluid layer provided $R_3 > \frac{\alpha_2}{\hat{\alpha}_2}$.

(v) The temperature is making the fluid flow stable when $R_3 < \frac{\alpha_2}{\hat{\alpha}_2}$.
(vi) Critical thermal Rayleigh numbers are obtained for varrying values of concentration gradient for different type of surfaces. It is found that critical Rayleigh number is minimum in the case when both the boundaries are free and maximum when both the boundaries are rigid. These critical values have been presented in table 1.

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